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1998 J. Phys. A: Math. Gen. 31 6717

(http://iopscience.iop.org/0305-4470/31/31/016)

J. Phys. A: Math. Gen. **31** (1998) 6717–6724. Printed in the UK PII: S0305-4470(98)92425-8

A generalized Duffin–Kemmer–Petiau oscillator

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Received 12 March 1998

Abstract. We propose a generic $S = 1$ relativisitic oscillator model which extends the class of relativistic bosonic oscillators. The Duffin–Kemmer–Petiau (DKP) oscillator we introduced in an earlier work can be recovered as an element of a family of DKP oscillators that can be realized within this generic model. We present the formalism for the exact quantum mechanical treatment of this generic model and, for illustration, compute the eigenvalues of a particular family of relativistic oscillators.

1. Introduction

There has been a recent revival of interest in the Duffin–Kemmer–Petiau (DKP) equation [1–3], a first-order relativistic equation for spin-0 and spin-1 particles, and its relevance to some problems in nuclear [4–7] and particle [8, 9] physics. In particular, as part of a wider research effort on relativistic quantum harmonic oscillators of particles of arbitrary spin and their properties [10–16], some interest has also focused on its associated relativistic oscillators of scalar and vector bosons [17–20].

In an earlier investigation [19] we introduced a new potential in the DKP equation. Because in the non-relativistic limit the spin-0 representation of this DKP equation yields the usual harmonic oscillator while its spin-1 representation leads to a harmonic oscillator with a spin–orbit coupling of the Thomas form, we called the system a DKP oscillator. This oscillator is a realizable relativistic generalization of the quantum harmonic oscillator for scalar and vector bosons.

The non-relativistic harmonic oscillator is quadratic in $\boldsymbol{r}$ but the Lorentz-tensor external interaction we introduced was constructed by a non-minimal substitution, linear in $\boldsymbol{r}$, into the free covariant DKP equation. This procedure is analogous to that used for the Dirac oscillator [11–13]. Physically, just as in the Dirac case [12, 13], this potential can be interpreted as representing a zero charge particle interacting, via its anomalous magnetic dipole moment, with a radial electric field.

However, as will be shown below, this DKP oscillator model is not sufficiently general since it actually involves only one of the two irreducible antisymmetric second-rank Lorentz-tensors that can be obtained from the Kemmer β matrices [21, 22]. The aim of this paper is to identify a generalized DKP oscillator model, constructed from these two independent antisymmetric tensors, and investigate the conditions under which classes of DKP oscillators can be realized.

This paper will also deal with the general quantum mechanical eigenproblem of families of $S = 1$ DKP oscillator systems. As an illustration, the *exact* eigenvalues of a particular class of oscillators will be calculated within this formalism. These are found to be different from those obtained for the original DKP oscillator [19].

2. Covariant form of the DKP oscillator

In an earlier work [19], we constructed the DKP oscillator system by operating the non-minimal substitution $\boldsymbol{p} \longrightarrow \boldsymbol{p} - \mathrm{i}m\omega\eta^0\boldsymbol{r}$, where ω is the oscillator frequency and $\eta^0 = 2\beta^{0^2} - 1$, into the relativistic DKP equation for a free scalar or vector boson of mass m, i.e.

$$(c\boldsymbol{\beta}\cdot(\boldsymbol{p} - \mathrm{i}m\omega\eta^0\boldsymbol{r}) + mc^2)\psi = \mathrm{i}\hbar\beta^0\frac{\partial\psi}{\partial t}. \tag{1}$$

The DKP algebra matrices β^μ ($\mu = 0, 1, 2, 3$) satisfy the commutation relation

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu. \tag{2}$$

This DKP oscillator system can be written in a covariant form a

$$\left(\beta^\mu p_\mu - m + \lambda\frac{e}{2m}\Sigma^{\mu\nu}F_{\mu\nu}\right)\psi = 0 \tag{3a}$$

where $\lambda = m^2\omega/e$ and $\Sigma^{\mu\nu}$ is the second-rank antisymmetric tensor

$$\Sigma^{\mu\nu} = \{\beta^0, \mathrm{i}[\beta^\mu, \beta^\nu]\} \tag{3b}$$

while $F_{\mu\nu}$ is the electromagnetic antisymmetric tensor

$$F_{\mu\nu} = u_\mu x_\nu - u_\nu x_\mu \tag{3c}$$

with the spacetime coordinate vector $x^\mu = (t, \boldsymbol{x})$ and the timelike unit vector $u_\mu = (1, \mathbf{0})$.

Physically, if $\Sigma^{\mu\nu}$ were to be construed as the physical spin tensor, the covariant equation (3*a*) would suggest that the DKP oscillator has the electromagnetic interpretation of a neutral particle interacting via its anomalous magnetic dipole moment λ with a radial electric field. This view would coincide with one of the alternative interpretations of the Dirac oscillator [12, 13, 23]. This interpretation is, however, not warranted since it is not clear whether $\Sigma^{\mu\nu}$ is the appropriate spin magnetic moment operator. The definitional ambiguities and difficulties with this operator in the DKP framework have already been discussed by Kemmer [2].

In equation (3*b*), the antisymmetric $\Sigma^{\mu\nu}$ tensor is just a particular linear combination of $\beta^0\sigma^{\mu\nu}$ and $\sigma^{\mu\nu}\beta^0$, more precisely the anticommutator of β^0 with the intrinsic spin tensor operator $\sigma^{\mu\nu} = \mathrm{i}[\beta^\mu, \beta^\nu]$. Using only the intrinsic spin tensor $\sigma^{\mu\nu}$ as an alternative antisymmetric tensor in equation (3*a*) can be shown not to generate an oscillator system and does not satisfy hermiticity requirements. For the expectation value of any DKP operator Q, considering linear combinations of $\beta^0 Q$ and $Q\beta^0$ arises out of general hermiticity constraints [2, 24, 25].

Now what is problematic with this DKP oscillator model is that $\Sigma^{\mu\nu}$ is not sufficiently general since it involves only one of the two independent antisymmetric second-rank Lorentz-tensors (i.e. only $\sigma^{\mu\nu}$) that can actually be obtained from the Kemmer β matrices. It omits couplings associated with the second independent antisymmetric tensor $\theta^{\mu\nu} = \{\beta^\mu\beta_\mu, \sigma^{\mu\nu}\}$ [21, 22].

A more general interaction should involve combinations of these two independent antisymmetric tensors. Here we shall consider general interactions built from

$$\Sigma^{\mu\nu} = \{\beta^0, g_1\sigma^{\mu\nu} + g_2\theta^{\mu\nu}\} \tag{4}$$

where g_1 and g_2 are some appropriate constants. This intuitive linear admixture draws *heuristic* support from studies on the consistency of spin-1 theories in external electromagnetic fields [21, 26] which show that accounting for anomalous electromagnetic interactions by constructing all possible antisymmetric tensors out of the Kemmer β matrices and coupling them appropriately to corresponding tensors constructed from the arbitrary electromagnetic field tensor $F_{\mu\nu}$, while preserving the causality of propagation and the reality of eigenvalues, restricts the possible forms of the interactions to

$$\mathcal{I}_{\text{anom.}} = \frac{\mathrm{i}e}{2m}(g_1\sigma^{\mu\nu} + g_2\theta^{\mu\nu})F_{\mu\nu}. \tag{5}$$

Furthermore, general interactions constructed from equation (4) would preserve to some extent the content of our basic DKP oscillator [19] as they would be reducible to it in some limit.

It remains to be seen, however, whether (and under which conditions) models constructed from these two independent antisymmetric tensors can actually realize $S = 1$ relativistic oscillator systems. We now address this issue.

3. A generalized DKP oscillator

The *necessary* criterion we shall adopt to establish whether a given DKP quantum oscillator model is an adequate $S = 1$ relativistic quantum oscillator consists of requiring that the usual three-dimensional (3D) oscillator should be recovered in the non-relativistic limit. There are no known *sufficient* criteria to determine uniquely relativistic generalizations of quantum oscillators [27].

Consider the heuristic generalized DKP oscillator model

$$(\beta^\mu p_\mu - m + \tfrac{1}{2}m\omega\Xi^{\mu\nu}F_{\mu\nu})\psi = 0 \tag{6}$$

where the antisymmetric $\Xi^{\mu\nu}$ tensor stands for the following combinations of β^0 with the $\sigma^{\mu\nu}$ and $\theta^{\mu\nu}$ tensors:

$$\Xi^{\mu\nu} = g_1^+\beta^0\sigma^{\mu\nu} + g_1^-\sigma^{\mu\nu}\beta^0 + g_2^+\beta^0\theta^{\mu\nu} + g_2^-\theta^{\mu\nu}\beta^0 \tag{7}$$

$g_1^\pm$ and $g_2^\pm$ being appropriate constants. In order to investigate whether, and under which conditions, this model meets the necessary criterion above, we now seek the non-relativistic limit of (6).

In the spin-1 representation of equation (6), the dynamical state ψ is chosen as the 10-component spinor

$$\psi(\boldsymbol{r}) = \begin{pmatrix} \mathrm{i}\varphi(\boldsymbol{r}) \\ \boldsymbol{A}(\boldsymbol{r}) \\ \boldsymbol{B}(\boldsymbol{r}) \\ \boldsymbol{C}(\boldsymbol{r}) \end{pmatrix} \qquad \text{with } \boldsymbol{A} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},\ \boldsymbol{B} \equiv \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},\ \text{and } \boldsymbol{C} \equiv \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \tag{8}$$

so that, for stationary states, the equation of motion equation (6) decomposes into

$$\begin{aligned} m\varphi &= \mathrm{i}(\boldsymbol{p} - \mathrm{i}(g_1^- + 6g_2^-)m\omega\boldsymbol{r})\cdot\boldsymbol{B} \\ m\boldsymbol{A} &= E\boldsymbol{B} - (\boldsymbol{p} + \mathrm{i}(g_1^+ + 4g_2^+)m\omega\boldsymbol{r})\wedge\boldsymbol{C} \\ m\boldsymbol{B} &= E\boldsymbol{A} + \mathrm{i}(\boldsymbol{p} + \mathrm{i}(g_1^+ + 6g_2^+)m\omega\boldsymbol{r})\varphi \\ m\boldsymbol{C} &= -(\boldsymbol{p} - \mathrm{i}(g_1^- + 4g_2^-)m\omega\boldsymbol{r})\wedge\boldsymbol{A}. \end{aligned} \tag{9}$$

Given that $\boldsymbol{A}$ is the 3-component spinor analogous to the Dirac upper component, the wave equation of interest to analyse the non-relativistic limit is the one satisfied by $\boldsymbol{A}$ [6, 28]. Using the definitions $\boldsymbol{p}^\pm = \boldsymbol{p} \pm \mathrm{i}(g_1^\pm + 6g_2^\pm)m\omega\boldsymbol{r}$ and $\boldsymbol{q}^\pm = \boldsymbol{p} \pm \mathrm{i}(g_1^\pm + 4g_2^\pm)m\omega\boldsymbol{r}$, one can eliminate φ, $\boldsymbol{B}$ and $\boldsymbol{C}$ in favour of $\boldsymbol{A}$ thus finding that

$$(E^2 - m^2)\boldsymbol{A} = \boldsymbol{p}^-(\boldsymbol{p}^+\cdot\boldsymbol{A}) - \boldsymbol{q}^-\wedge(\boldsymbol{q}^+\wedge\boldsymbol{A}) - \frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^-\cdot[\boldsymbol{q}^-\wedge(\boldsymbol{q}^+\wedge\boldsymbol{A})]\}. \tag{10}$$

In the non-relativistic limit, the third term in equation (10) becomes negligible, since it is of order $1/m^3$. Using relevant vector algebra and spin identities, the *exact* calculation of the first two terms on the right-hand side of equation (10) leads to

$$\begin{aligned}(E^2 - m^2)\boldsymbol{A} \simeq {} & [\boldsymbol{p}^2 + (a_+a_-)m^2\omega^2\boldsymbol{r}^2 + (3a_+ - 2a_- + 2b_-)m\omega]\boldsymbol{A} \\ & +[(a_+ + b_-)m\omega\boldsymbol{L}\cdot\boldsymbol{s} + (a_+ - a_-)m\omega\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{p})]\boldsymbol{A} \\ & +[(a_- - a_+ - b_- + b_+)m\omega\mathrm{i}(\boldsymbol{s}\cdot\boldsymbol{p})(\boldsymbol{s}\cdot\boldsymbol{r}) \\ & +(b_+b_- - a_-a_+)m^2\omega^2(\boldsymbol{s}\cdot\boldsymbol{r})(\boldsymbol{s}\cdot\boldsymbol{r})]\boldsymbol{A} \qquad (11)\end{aligned}$$

where $\boldsymbol{L}$ is the orbital angular momentum while $\boldsymbol{s}$ is the 3×3 spin-1 operator in the $(s_m)_{kl} = -\mathrm{i}\varepsilon_{klm}$ representation, ε_{klm} being the totally antisymmetric Levi-Cevita symbol. The coefficients $a_\pm$ and $b_\pm$ stand for $g_1^\pm + 6g_2^\pm$ and $g_1^\pm + 4g_2^\pm$ respectively . This Schrödinger-equivalent equation contains the usual 3D oscillator potential with a non-vanishing zero-point motion energy, in addition to a spin–orbit coupling and a non-local Darwin term respectively shown in the second term of equation (11). The third contribution is a sum of two among the three possible tensor potentials built up from $\boldsymbol{s}$, $\boldsymbol{r}$ and $\boldsymbol{p}$ and usually encountered in spin-1 nuclear dynamics [29]; they arise here with constant form factors.

In order to meet the requirement for an adequate relativistic generalization for $S = 1$ quantum oscillators stated above, the form factors of the Darwin and tensor potentials should be constrained to zero. This restricts the values of the coefficients to

$$a_+ = a_- \qquad b_+ = b_- \qquad \text{and} \qquad b_+b_- = a_-a_+. \qquad (12)$$

Equations (6) and (7) along with this condition fully specify what we shall denote the generalized DKP oscillator model.

Note that in the particular case where $a_\pm = 1$ and $b_\pm = 1$, with $g_1^\pm = 1$ and $g_2^\pm = 0$, this generalized oscillator simply reduces to the basic DKP oscillator [19] ($\Xi^{\mu\nu} = \{\beta^0, \sigma^{\mu\nu}\}$). The non-relativistic limit of the Schrödinger-equivalent equation (11) ($E = \varepsilon + m$ with $\varepsilon \ll mc$) yields

$$\varepsilon\boldsymbol{A} \simeq \left[\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\omega - \omega\boldsymbol{L}\cdot\boldsymbol{s}\right]\boldsymbol{A} \qquad (13)$$

i.e. the usual harmonic oscillator in addition to a spin–orbit coupling with the same sign, but half the strength, as the one obtained from the Dirac oscillator [11–13].

In general though, various classes of other oscillator models with alternative oscillator and spin–orbit coupling strengths, and zero-point energies in the non-relativistic limit can be constructed from this generic DKP oscillator system. We now consider their quantum mechanical treatment by solving the eigen-problem of the generic DKP oscillator model.

4. Eigen-solutions of the generalized DKP oscillator

For the calculation of the *exact* solution to the generalized DKP oscillator (6) eigen-problem, the general form of the DKP eigenfunctions we use take the form [28]

$$\psi_{JM}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} \mathrm{i}\phi_{nJ}(r)Y_{JM}(\Omega) \\ \sum_L F_{nJL}(r)\boldsymbol{Y}^M_{JL1}(\Omega) \\ \sum_L G_{nJL}(r)\boldsymbol{Y}^M_{JL1}(\Omega) \\ \sum_L H_{nJL}(r)\boldsymbol{Y}^M_{JL1}(\Omega) \end{pmatrix} \qquad (14)$$

where F_{nJL}, G_{nJL} and H_{nJL} are radial wavefunctions, $Y_{JM}(\Omega)$ are the spherical harmonics of order J and $\boldsymbol{Y}^M_{JL1}(\Omega)$ represent the normalized vector spherical harmonics. Inserting ψ_{JM}

into equation (6) and using relevant properties of spherical harmonics eliminates the spin-angular functions and yields 10 coupled radial differential equations [28]. These decouple into two disjoint sets associated with opposite parities, the $(-1)^J$ solutions pertaining to natural-parity (or magnetic-like) states and the $(-1)^{J+1}$ solutions referred to as unnatural-parity (or electric-like) states. With the definition

$$R_{nJJ}(r) = R_0 \qquad R_{nJJ\pm1}(r) = R_{\pm1} \qquad R \equiv F, G, H \tag{15}$$

and setting $\alpha_J = \sqrt{(J+1)/(2J+1)}$ and $\zeta_J = \sqrt{J/(2J+1)}$, the radial differential equations associated with $(-1)^J$ parity are

$$EF_0 = mG_0 \tag{15a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} + b_- m\omega r\right) F_0 = -\frac{1}{\zeta_J} mH_1 \tag{15b}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} + b_- m\omega r\right) F_0 = -\frac{1}{\alpha_J} mH_{-1} \tag{15c}$$

$$\zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} - b_+ m\omega r\right) H_1 \alpha_J \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} - b_+ m\omega r\right) H_{-1} = (EG_0 - mF_0). \tag{15d}$$

The radial differential equations associated with unnatural parity states are coupled in such a way that

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} - b_+ m\omega r\right) H_0 = -\frac{1}{\zeta_J}(mF_1 - EG_1) \tag{16a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} - b_+ m\omega r\right) H_0 = -\frac{1}{\alpha_J}(mF_{-1} - EG_{-1}) \tag{16b}$$

$$-\zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} + b_- m\omega r\right) F_1 - \alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} + b_- m\omega r\right) F_{-1} = mH_0 \tag{16c}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} - \frac{a_+ m\omega r}{\hbar}\right)\phi = \frac{1}{\alpha_J}(EF_1 - mG_1) \tag{16d}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} - a_+ m\omega r\right)\phi = -\frac{1}{\zeta_J}(EF_{-1} - mG_{-1}) \tag{16e}$$

$$-\alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} + a_- m\omega r\right) G_1 + \zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} + a_- m\omega r\right) G_{-1} = m\phi. \tag{16f}$$

Of course, for relativistic oscillator eigen-solutions to obtain, these sets of differential equations have to be solved keeping the coefficients $a_\pm$ and $b_\pm$ restricted to satisfy equation (12).

For illustrating how the exact analytic solutions of equations (15) and (16) can be found, we now look at the particular family of oscillators such that $a_\pm = -1$ and $b_\pm = -1$.

The exact solution for the magnetic-like states is obtained by eliminating G_0, $H_{\pm1}$ in equation (15*d*) so that one gets the radial equation for the 3D harmonic oscillator

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + E^2 - m^2 - m\omega - m^2\omega^2 r^2 - \frac{J(J+1)}{r^2}\right) F_0(r) = 0 \tag{17}$$

whose associated eigenvalues are simply the degenerate and equally spaced

$$\frac{1}{2m}(E_{N,J}^2 - m^2) = (N+2)\omega \tag{18}$$

where N is the principal quantum number defined as $N = 2n + J$, n representing the radial quantum number.

For unnatural parity states, using the same technique as in [19] transforms the problem of solving equations (16a–f) to that of solving

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (E^2 - m^2) + 3m\omega - m^2\omega^2 r^2 - \frac{J(J+1)}{r^2}\right)\phi = 2\sqrt{J(J+1)}E\omega H_0 \tag{19a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (E^2 - m^2) + m\omega - m^2\omega^2 r^2 - \frac{J(J+1)}{r^2}\right)H_0 = 2\sqrt{J(J+1)}E\omega\phi \tag{19b}$$

$$\begin{pmatrix} F_1 \\ G_1 \end{pmatrix} = \frac{1}{E^2 - m^2}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} + m\omega r\right)\begin{pmatrix} \alpha_J E & \zeta_J m \\ \alpha_J m & \zeta_J E \end{pmatrix}\begin{pmatrix} \phi \\ H_0 \end{pmatrix} \tag{19c}$$

$$\begin{pmatrix} F_{-1} \\ G_{-1} \end{pmatrix} = \frac{1}{E^2 - m^2}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} + m\omega r\right)\begin{pmatrix} -\zeta_J E & \alpha_J m \\ -\zeta_J m & \alpha_J E \end{pmatrix}\begin{pmatrix} \phi \\ H_0 \end{pmatrix}. \tag{19d}$$

Implementing the diagonalization procedure

$$\begin{pmatrix} \phi \\ H_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\gamma & \kappa \\ \kappa & -1-\gamma \end{pmatrix}\begin{pmatrix} R_+ \\ R_- \end{pmatrix} \qquad \text{with } \gamma = \sqrt{1+\kappa^2} \text{ and } \kappa = 2\sqrt{J(J+1)}\frac{E}{m} \tag{20}$$

leads to the decoupling of equations (19a, b) into the following 3D oscillator-like radial equations

$$\frac{\mathrm{d}^2 R_+}{\mathrm{d}r^2} + \left(E^2 - m^2 + 2m\omega - \omega\sqrt{m^2 + 4J(J+1)E^2} - m^2\omega^2 r^2 - \frac{J(J+1)}{r^2}\right)R_+ = 0 \tag{21a}$$

$$\frac{\mathrm{d}^2 R_-}{\mathrm{d}r^2} + \left(E^2 - m^2 + 2m\omega + \omega\sqrt{m^2 + 4J(J+1)E^2} - m^2\omega^2 r^2 - \frac{J(J+1)}{r^2}\right)R_- = 0. \tag{21b}$$

The eigenfunctions R_+ and R_- are orthogonal and for $J = 0$ reduce to the radial functions ϕ and $-H_0$ respectively. The eigenvalue E_+ of equation (21a) simply satisfies

$$(E_+^2 - m^2) - \omega\sqrt{m^2 + 4J(J+1)E_+^2} = (2N-1)\omega m \tag{22a}$$

while the eigenvalue E_- of equation (21b) is given by

$$(E_-^2 - m^2) + \omega\sqrt{m^2 + 4J(J+1)E_-^2} = (2N-1)\omega m \tag{22b}$$

where the principal quantum number N is a positive integer. The nonlinear eigenvalue equations (22a, b) can be solved to yield

$$\frac{1}{2m}(E_\pm^2 - m^2) = \left(N - \frac{1}{2}\right)\omega + J(J+1)\frac{\omega^2}{m} \pm \Delta \tag{23a}$$

for which

$$\Delta = \omega\left(J + \frac{1}{2}\right)\left(1 + \frac{a_1}{a_0}\frac{\omega}{m} + \frac{a_2}{a_0}\left(\frac{\omega}{m}\right)^2\right)^{\frac{1}{2}} \tag{23b}$$

with $a_0 = (2J+1)^2$, $a_1 = 4J(J+1)(2N-1)$ and $a_2 = 4J^2(J+1)^2$.

In unnatural parity states, the energy of this relativistic oscillator system involves the usual 3D harmonic oscillator energy (with an $1\hbar\omega$ decrease in the zero-point energy), a rotational energy term proportional to $J(J+1)$ and an energy contribution Δ with no simple physical interpretation.

This result can be verified by taking its non-relativisic limits and comparing them with those obtained directly from equation (11). In the limit where the oscillator frequencies are such that $\hbar\omega \ll mc^2$, keeping only the first-order term in ω in equations (22a, b) yields

$$\frac{1}{2mc^2}(E_+^2 - m^2c^4) \equiv \epsilon_{\text{n.r.}}^+ \simeq (N + J)\omega \qquad (24a)$$

$$\frac{1}{2mc^2}(E_-^2 - m^2c^4) \equiv \epsilon_{\text{n.r.}}^- \simeq (N - J - 1)\omega \qquad (24b)$$

in agreement with the eigenvalues of equation (13) which accounts for a 3D non-relativistic oscillator with a spin–orbit coupling of strength $+\hbar\omega$.

Compared with the unnatural parity eigenspectra which we found for the basic DKP oscillator [19], the spectra of this family of oscillators differ in the magnitude of the zero-point motion energy and in the order of the spin–orbit splittings which are inverted relative to each other.

Altogether, the formalism presented here allows the general treatment of classes of relativistic DKP oscillators, each of which is specified by a particular choice of the coefficients $a_\pm$ and $b_\pm$. We have illustrated how exact analytic eigenspectra can be derived for a particular set of parameters.

It is not the case, however, that *exact* analytic solutions can be found for all of them. While one can solve exactly the natural parity eigenstate problem in all cases, for unnatural parity states exact solutions exist only for particular sets of $a_\pm$ and $b_\pm$.

5. Conclusion

We have introduced a generalized DKP oscillator by a procedure in which the two independent antisymmetric second-rank Lorentz tensors, constructed from Kemmer β matrices, are combined. This generic model extends the class of relativistic bosonic oscillators while preserving the content of, and in some limit being reducible to, the particular DKP oscillator we discussed earlier [19].

In the non-relativistic limit, the DKP equation of motion of our generic model leads to the usual harmonic oscillator with a spin–orbit coupling of the Thomas form in addition to a Darwin and two tensor potentials with constant form factors. The adequacy requirement for the relativistic generalization of quantum oscillators prescribes specific constraints on the parameters of the model.

We have given the formalism for the exact quantum mechanical treatment of the generic model and, for illustration, computed the eigenvalues of a particular family of relativistic oscillators.

Acknowledgment

The financial support of EPSRC through GR/J95867 is gratefully acknowledged.

References

[1] Petiau G 1936 *Acad. R. Belg. Cl. Sci. Mem. Collect.* **16** no 2
Duffin R J 1939 *Phys. Rev.* **54** 1114

[2] Kemmer N 1938 *Proc. R. Soc.* **166** 127
Kemmer N 1939 *Proc. R. Soc.* A **173** 91

[3] Krajcik R A and Nieto M M 1977 *Am. J. Phys.* **45** 818
Fischbach E, Nieto M M and Scott C K 1972 *Prog. Theor. Phys.* **48** 574

[4] Kälbermann G 1986 *Phys. Rev.* C **34** 2240
[5] Clark B C, Hama S, Kälbermann G, Mercer R L and Ray L 1985 *Phys. Rev. Lett.* **55** 592
[6] Kozack R E, Clark B C, Hama S, Mishra V K, Mercer R L and Ray L 1989 *Phys. Rev.* C **40** 2181
[7] Ait-Tahar S, Al-Khalili J S and Nedjadi Y 1995 *Nucl. Phys.* A **589** 307
[8] Daicic J and Frankel N E 1992 *Prog. Theor. Phys.* **88** 1
[9] Beckers J, Debergh N and Nikitin A G 1995 *Fortschr. Phys.* **43** 81
[10] Moshinsky M and del Sol Mesa A 1996 *J. Phys. A: Math. Gen.* **29** 4217
[11] Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[12] Moreno M and Zentella A 1989 *J. Phys. A: Math. Gen.* **22** L821
[13] Benitez J, Martinez y Romero R P, Nunez-Yepez H N and Salas-Brito A L 1990 *Phys. Rev. Lett.* **64** 1643
Martinez y Romero R P, Nunez-Yepez H N and Salas-Brito A L 1995 *Eur. J. Phys.* **16** 135
[14] Moshinsky M, Loyala G and Villegas C 1991 *J. Math. Phys.* **32** 373
[15] Moreno M, Martinez R and Zentella A 1990 *Mod. Phys. Lett.* A **5** 949
[16] Dvoeglazov V V 1994 *Nuovo Cimento* A **107** 1785
Dvoeglazov V V 1996 *Rev. Mex. Fis.* **42** 172
[17] Debergh N, Ndimubandi J and Strivay D 1992 *Z. Phys.* C **56** 421
[18] Beckers J and Debergh N 1992 *J. Math. Phys.* **33** 3387
[19] Nedjadi Y and Barrett R C 1994 *J. Phys. A: Math. Gen.* **27** 4301
[20] Nedjadi Y, Ait-Tahar S and Barrett R C 1998 *J. Phys. A: Math. Gen.* **31** 3867
[21] Vijayalakshmi B, Seetharaman M and Mathews P M 1979 *J. Phys. A: Math. Gen.* **12** 665
[22] Glass A S 1971 *Commun. Math. Phys.* **23** 176
[23] Crawford J P 1993 *J. Math. Phys.* **34** 4428
[24] Heitler W 1943 *Proc. R. Irish Acad.* A **49** 1
[25] Krajcik R A and Nieto M M 1974 *Phys. Rev.* D **10** 4049
Krajcik R A and Nieto M M 1975 *Phys. Rev.* D **11** 1459
[26] Daicic J and Frankel N E 1993 *J. Phys. A: Math. Gen.* **26** 1397 and references therein
[27] Magnollay P 1985 Quantum relativistic rotators and oscillators *PhD Thesis* University of Michigan
[28] Nedjadi Y and Barrett R C 1994 *J. Math. Phys.* **35** 4517
[29] Satchler G R 1960 *Nucl. Phys.* **21** 116